

On product spacetime with 2-sphere of constant curvature

Naresh Dadhich*

*Inter-University Centre for Astronomy & Astrophysics,
Post Bag 4, Ganeshkhind, Pune - 411 007, India.*

Abstract

If we consider the spacetime manifold as product of a constant curvature 2-sphere (hypersphere) and a 2-space, then solution of the Einstein equation requires that the latter must also be of constant curvature. There exist only two solutions for classical matter distribution which are given by the Nariai (anti) metric describing an Einstein space and the Bertotti - Robinson (anti) metric describing a uniform electric field. These two solutions are transformable into each other by letting the timelike convergence density change sign. The hyperspherical solution is anti of the spherical one and the vice -versa. For non classical matter, we however find a new solution, which is electrograv dual to the flat space, and describes a cloud of string dust of uniform energy density. We also discuss some interesting features of the particle motion in the Bertotti - Robinson metric.

PACS numbers : 04.20,04.60,98.80Hw

*E-mail : nkd@iucaa.ernet.in

We consider the 4-dimensional spacetime as a product manifold of two spaces $R^2 \times S^2$ with S^2 having constant curvature. The latter condition implies that $R_2^2 = R_3^3 = \text{const.}$ and $R_0^0 = R_1^1$ where the coordinates are designated as $t = 0, z = 1, \theta = 2, \varphi = 3$. Each space is a 2-space intersecting the other orthogonally and hence each would be specified by a single curvature, which would be R^{23}_{23} for S^2 and R^{01}_{01} for R^2 . The Einstein equation would hence be a relation between them and it turns out that the only classical matter distribution it can sustain is the Einstein space ($R_{ab} = \Lambda g_{ab}, \rho + p = 0$) or the uniform electric field, with $R = 0$.

Let us define the energy density by $\rho = T_{ab}u^au^b$, timelike convergence density by $\rho_t = (T_{ab} - 1/2 T g_{ab})u^au^b$ and the null convergence density by $\rho_n = T_{ab}k^ak^b$ where $u^au_a = 1, k^ak_a = 0$ [1]. Note that for the product manifold, $\rho = R_2^2, \rho_n = R_0^0 - R_1^1 = 0, \rho_t = -R_0^0$. For the classical matter distribution, there exist the only two possibilities; (i) $\rho + \rho_t = 0$, the Einstein space given by the Nariai metric [2] and (ii) $\rho - \rho_t = 0$, the uniform electric field given by the Bertotti - Robinson metric [3,4], and $\rho_n = 0$ always. The only other possibility could be of the two curvatures one could vanish; i.e. either $\rho = 0$ or $\rho_t = 0$. The former cannot vanish because it defines the constant curvature of the 2-sphere (hypersphere), the construction we began with. Even for the non classical matter, the only possibility is $\rho_t = 0$, which is the equation of state for the cosmological defects, string dust, global monopole and global texture [5]. The new solution that we shall obtain could describe a string dust of constant energy density. The solution could be shown to be the electrograv dual to the flat space [6,7].

For defining the electrogravity duality, we decompose, in analogy with the Maxwell theory, the Riemann curvature into electric and magnetic parts relative to a timelike unit vector. Since the Riemann curvature is a double 2-form, and hence it would always be the double projection. The active electric part is given by $E_{ab} = R_{acbd}u^cu^d$, the passive electric part by $\tilde{E}_{ab} = *R * acbd u^cu^d$ and the magnetic part by $H_{ab} = *R_{acbd}u^cu^d$ where $*R * abcd = 1/4 \eta_{abmn} \eta_{cdpq} R^{mnpq}$ and η_{abcd} is the 4-volume element. In terms of the electromagnetic parts, the Ricci curvature is given by

$$R_{ab} = E_{ab} + \tilde{E}_{ab} + (E + \tilde{E})u_a u_b - \tilde{E}g_{ab} + u^c H^{mn}(\eta_{acmn}u_b + \eta_{bcmn}u_a) \quad (1)$$

where $E = E_a^a, \tilde{E}_a^a = \tilde{E}$, and $\rho = -\tilde{E}, \rho_t = -E$.

By the electrogravity duality we mean [6],

$$E_{ab} \leftrightarrow \tilde{E}_{ab}, H_{ab} \rightarrow H_{ab} \quad (2)$$

That is the interchange of active and passive electric parts. Note that under duality, the Ricci and the Einstein tensors interchange, which is because contraction of the Riemann is Ricci while its double dual is Einstein.

Clearly under the duality transformation, $\rho \leftrightarrow \rho_t, \rho_n \rightarrow \rho_n$ which means the Einstein space would be anti dual while the uniform electric field solution would be self dual. We would further show that the new solution describing a string dust would be dual to the flat space.

We write the metric for the product spacetime as

$$ds^2 = c^2 dt^2 - a^2 dz^2 - \frac{1}{\lambda^2} (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (3)$$

where λ is a constant with dimension of inverse length, and c, a are in general functions of z and t which run from $-\infty$ to ∞ . There are only two independent components of the Ricci tensor which read as

$$-\rho_t = R_0^0 = R_1^1 = \frac{1}{c^2} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}\dot{c}}{ac} \right) - \frac{1}{a^2} \left(\frac{c''}{c} - \frac{a'c'}{ac} \right), \quad \rho = R_2^2 = R_3^3 = \lambda^2 \quad (4)$$

The two classical matter solutions would be obtained by solving the single equation $\rho = \lambda^2 = \pm \rho_t$. For its solution we will have to choose the one of a and c or a relation between them. There could exist both static and non static solutions but one could always be transformed into the other. We shall give the both in the gauge $ac = 1$. The solutions are given as follows.

I. $\rho + \rho_t = 0$: The Nariai metric [2] for the Einstein space,

$$c^2 = a^{-2} = (1 + \lambda^2 t^2)^{-1}, \quad 1 - \lambda^2 z^2 \quad (5)$$

which gives $\rho = \lambda^2 = -\rho_t$.

II. $\rho = \rho_t = \lambda^2$: The Bertotti - Robinson metric [3,4] for the uniform electric field,

$$c^2 = a^{-2} = (1 - \lambda^2 t^2)^{-1}, \quad 1 + \lambda^2 z^2 \quad (6)$$

The electric field would be along the z -axis, $F_{01} = \pm \sqrt{2}ac\lambda$, which would imply

$$(F^{ij} \sqrt{-g})_{,j} = 0 \quad (7)$$

where a coma denotes ordinary derivative. Thus the electric field is uniform and is given by $\pm\sqrt{2}\lambda$. This equation indicates absence of charges, which would be lying at $z = \pm\infty$ to produce a uniform field.

Let us note some of the interesting properties of the two solutions:

(a) $N \leftrightarrow BR$ when $\rho_t \leftrightarrow -\rho_t$, where N stands for the Nariai metric and BR for the Bertotti - Robinson metric. That means changing the sign of λ^2 only in c .

(b) The anti metrics of N and BR would be given by letting sphere ($\sin\theta$) to go to hypersphere ($\sinh\theta$) in the metric. That is, in N and BR let both $\rho \rightarrow -\rho, \rho_t \rightarrow -\rho_t$ to get to their anti counterparts.

(c) The anti metrics are transformable into each-other as in (a) above by letting $\rho_t \leftrightarrow -\rho_t$.

(d) Under the duality transformation, $\rho \leftrightarrow \rho_t$, the Einstein space is anti dual, and the Nariai and anti Nariai are dual of each - other. On the other hand, the trace free matter field is self dual, and the BR and anti BR are dual of each other with both ρ and ρ_t changing sign. This is because under the duality gravitational constant changes sign [8].

Now let us turn to the only remaining possibility of a curved spacetime for the metric (3), e.g. $\rho_t = -R_0^0 = 0$. That is R^2 is flat which would mean $c = a = 1$. The metric (3) would still be curved giving rise to the stresses $\rho = T_0^0 = T_1^1 = \lambda^2$ and the rest being zero. The equation of state, $\rho_t = 0$ is characteristic of topological defects; string dust, global monopole and global texture [5]. In the present case it is the string dust [9] that has constant energy density. The anti string dust metric would result if we let $\sin\theta \rightarrow \sinh\theta$, sphere \rightarrow hypersphere in the metric (3) with $a = c = 1$. This will have $\rho = -\lambda^2$.

The string dust solution can be obtained from the equation,

$$E_{ab} = 0, \quad \tilde{E}_b^a = -\lambda^2 g_1^a g_b^1 \quad (8)$$

which is dual to the effective flat space equation,

$$\tilde{E}_{ab} = 0, \quad E_b^a = -\lambda^2 g_1^a g_b^1 \quad (9)$$

for the metric (3). For a general spherically symmetric metric (replace z by r , $1/\lambda$ by b and take a, b, c as functions of r and t) it can be shown after considerable manipulation [10], that the above equation characterizes flat

space. This shows that the string dust which is the solution of the dual set (8) is dual to flat space which is the solution of the set (9).

The stress tensor for string dust is given by $T^{ab} = \epsilon \sigma^{ac} \sigma_c^b / -\gamma^{1/2}$ where ϵ is the proper energy density of the string cloud, γ_{ab} is the 2-dimensional metric on the string world sheet, σ^{ab} is the bivector associated with the world sheet: $\sigma^{ab} = \epsilon^{AB} \partial x^a / \partial \lambda^A \partial x^b / \partial \lambda^B$. Here ϵ^{AB} is the 2-D Levi-Civita tensor ($\epsilon^{01} = -\epsilon^{10} = 1$) and $\lambda^A = (\lambda^0, \lambda^1)$, where λ^0 and λ^1 are timelike and spacelike parameters on the string world sheet. In Ref.[9] the stress tensor for a spherically symmetric metric has been computed and shown that it has $T_0^0 = T_1^1 \propto 1/r^2$ which in our case would be proportional to a constant because the 2-sphere (hypersphere) has constant curvature. The matter distribution represented by the solution is indeed a cloud of string dust. As in other cases its anti version would have $\sin\theta$ being replaced by $\sinh\theta$, and $\rho \rightarrow -\rho$.

We shall next consider some interesting features of particle motion in the Bertotti - Robinson metric as given by the following form,

$$ds^2 = (1 + \lambda^2 z^2) dt^2 - (1 + \lambda^2 z^2)^{-1} dz^2 - \frac{1}{\lambda^2} (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (10)$$

Here the red-shifted proper acceleration is given by $-\lambda z$ which would be attractive/repulsive for $z > 0$ ($z < 0$). The most pertinent motion in this spacetime is the z -motion and it would clearly be simple harmonic about the stable state of rest $z = 0$. A particle sitting at $z = 0$, which could be chosen anywhere on the axis freely, would remain stay put there for ever. That would mean that a particle at rest would always remain at rest at any z while one in motion would execute simple harmonic oscillation about some appropriate $z = 0$ location. This happens because as it moves on the positive z , electrostatic energy lying behind it keeps on building up to pull it back, then it turns back and the same happens on the other side. This is an interesting simple harmonic oscillator which can be set up anywhere and is entirely maintained for ever by gravity (Fig. 1). Of course the catch is that the total energy of the system is infinite. It is though not very realistic, yet it is an interesting and novel example of a relativistic analogue of a simple classical situation.

What would happen if we consider motion of a charged particle in this spacetime? The effective potential for the pertinent motion would then be

given by

$$V = q\lambda z + (1 + \lambda^2 z^2)^{1/2} \quad (11)$$

where q is the charge per unit mass. Without loss of generality, we can take $q \geq 0$, because negative q would only imply reflection in z . Here there would be three cases corresponding to $q^2 < 1$, $q^2 > 1$ and $q^2 = 1$.

Case (i): For $q^2 < 1$, the potential would have the minimum at $\lambda z = -q/\sqrt{1 - q^2}$ and $V_{min} = \sqrt{1 - q^2}$. Particle would have oscillatory motion like the neutral particle with V_{min} being lowered and its location shifted on the negative z -axis. However the potential would not be symmetric but would be wider on the left of the minimum (Fig. 1). A particle sitting at the minimum of the potential would remain so for ever. That is, a charged particle in a uniform electric field is being kept at stable state of rest by gravity.

Case (ii): For $q^2 > 1$, there will be no minimum and $V = 0$ at $\lambda z = -1/\sqrt{q^2 - 1}$. It increases monotonically from negative large values to positive large values (Fig. 1). Negative energy orbits were first encountered in the Kerr spacetime [11] and they were confined to ergosphere with its outer boundary at $r = 2M$. By bringing in electromagnetic interaction, the effective ergosphere could be extended but its extent would always remain bounded [12]. In here we have a situation where occurrence of negative energy extends upto infinity in the negative z direction. Particles with both positive and negative energy can have a bounce and go back to infinity. This unphysical feature is due to infinite energy of the spacetime.

Case (iii): For $q^2 = 1$, there again occurs no minimum and it asymptotically tends to zero for negative z (Fig. 1).

With the constant curvature sphere (hypersphere) as one part of the product space, there are only three possible solutions. The two are the known solutions; Nariai cosmological metric for the Einstein space representing a fluid with $\rho + p = 0$ and the Bertotti - Robinson metric for uniform electric field. The third one is new and represents non classical matter of string dust of constant energy density. The Nariai metric has six parameter symmetry group and is not conformally flat. It is shown to be created by quantum polarization of vacuum and asymptotically it decays into the de Sitter and the Kasner spacetimes [13,14]. The Bertotti - Robinson metric is conformally flat and has also been used to calculate one-loop quantum gravitational corrections induced by fields of large mass [15]. It would be worthwhile to carry out

such calculations for the new string dust solution, which could in a way be viewed as "minimally" curved because it is completely free of the Newtonian gravity [7].

Apart from the new solution, the novel feature of the paper is to expose the inter - transformability of the Nariai and the Bertotti - Robinson metrics, the role of electrogravity duality in finding the new solution and some interesting features of particle motion in the spacetime of uniform electric field. It is interesting that a particle at rest in this spacetime would remain at rest for ever while the one in motion along the axis of the field would execute simple harmonic oscillation for ever.

Acknowledgement: I wish to thank Varun Sahni and Ramseh Tikekar for enlightening and useful discussions, and Ivor Robinson for reading the manuscript.

References

- [1] N. Dadhich, Spherically symmetric empty space and its dual in general relativity, to appear in Current Science, gr-qc/0003018.
- [2] H. Nariai (1950) Sci. Rep. Tohoku Univ. **34**, 160.
- [3] B. Bertotti (1959) Phys. Rev. **116**, 1331.
- [4] I. Robinson (1959) Bull. Acad. Pol. Sci. **7**, 351.
- [5] A. Vilenkin and E. P. S. Shellard (1994) *Cosmic Strings and Other Topological Defects* (Cambridge Univ. Press).
- [6] N. Dadhich (1999) Mod. Phys. Lett. **A14**, 337.
- [7] N. Dadhich, Electromagnetic duality in general relativity, gr-qc/9909067, to appear in GRG, June, 2000; the Ellis Festschrift number.
- [8] N. Dadhich (1999) Mod. Phys. Lett. **A14**, 759.
- [9] P. S. Letelier (1979) Phys. Rev. **D20**, 1294.
- [10] N. Dadhich (1970) Ph.D. Thesis, Pune University, unpublished.
- [11] R. Penrose (1969) Riv. Nuovo Cimento **1**, 252.
- [12] S. M. Wagh and N. Dadhich (1989) Physics Reports **183**, 137.
- [13] L. A. Kofman, V. Sahni and A. A. Starobinskii (1983) Sov. Phys. JETP **58**, 1090.
- [14] V. Sahni and L. A. Kofman (1986) Phys. Lett. **A117**, 275.
- [15] L. A. Kofman and V. Sahni (1983) Phys. Lett. **B127**, 197.

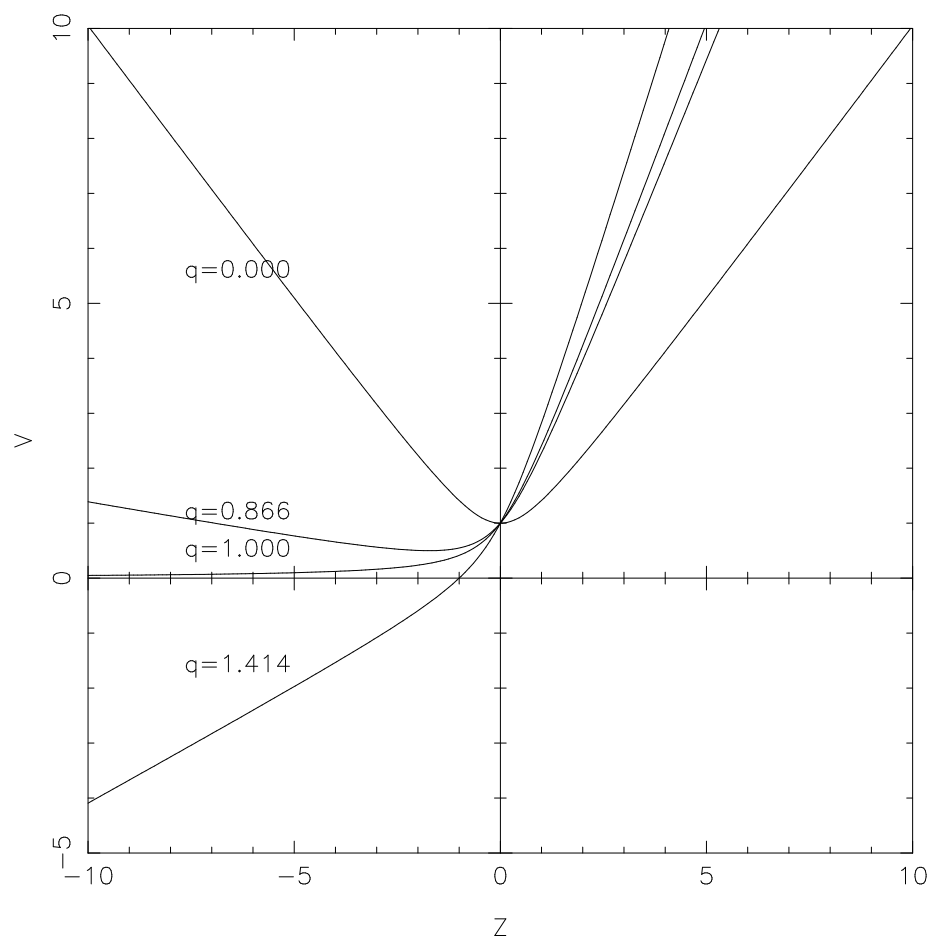


Figure 1: Potential plots for various values of q